British Mathematical Olympiad

Round 2 : Thursday, 27 February 1997

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (10-13 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Mar del Plata, Argentina, 21-31 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in late June or early July before leaving for Argentina.

Do not turn over until **told to do so.**

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1. Let M and N be two 9-digit positive integers with the property that if **any** one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

   Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

   Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

3. Find the number of polynomials of degree 5 with **distinct** coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

4. The set $S = \{1/r : r = 1, 2, 3, \ldots\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $1/20, 1/8, 1/5$ is such a progression, of length 3 (and common difference $3/40$). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and $11/40$ not being members of S).

   (i) Find a maximal progression in S of length 1996.

   (ii) Is there a maximal progression in S of length 1997?